Assignment 7

1. Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} nx & 0 \leqslant x \leqslant \frac{1}{n} \\ 1 & \frac{1}{n} \leqslant x \leqslant 1 \end{cases}$$

for $n = 1, 2, \cdots$. Check if the family $\{f_n\}$ is equicontinuous or not.

2. Let $\mathcal{F}$ be a family of equicontinuous functions on $[0,1]$. Suppose for every $x \in [0,1]$, the set $\{f(x) \mid f \in \mathcal{F}\}$ is bounded in $\mathbb{C}$. Show that $\mathcal{F}$ is bounded in $\mathcal{C}[0,1]$.

3. Suppose f is a real continuous function on $\mathbb{R}$. Define $f_n(x) = f(nx)$ for $n = 1, 2, \cdots$. If the family $\{f_n\}$ is equicontinuous on $[0,1]$, show that f is constant.

4. Show that a monotone function on $\mathbb{R}$ is continuous except possibly at a countable number of points.

5. Suppose $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ such that $|f_n(x)| < 1$ for all x and all n. Show that there exists a function f and a subsequence $\{n_k\}$ of $\{n\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x), \text{ for every } x \in \mathbb{R}.$$